

Locally standard torus actions and sheaves over Buchsbaum posets

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ABSTRACT. We consider a sheaf of exterior algebras on a simplicial poset S and introduce a notion of homological characteristic function. Two objects are associated with these data: a graded sheaf \mathcal{I} and a graded cosheaf $\hat{\Pi}$. When S is a homology manifold, we prove the isomorphism $H^{n-1-p}(S; \mathcal{I}) \cong H_p(S; \hat{\Pi})$ which can be considered as an extension of the Poincaré duality. In general, there is a spectral sequence $E_{p,q}^2 \cong H^{n-1-p}(S; \mathcal{U}_{n-1+q} \otimes \mathcal{I}) \Rightarrow H_{p+q}(S; \hat{\Pi})$, where \mathcal{U}_* is the local homology stack on S . This spectral sequence, in turn, extends Zeeman's spectral sequence in interpretation of McCrory. We apply these results to toric topology. Let X be an orientable manifold with locally standard action of a compact torus and acyclic proper faces of the orbit space. A principal torus bundle Y is associated with X and the orbit type filtration on X is covered by a topological filtration on Y . Then the second pages of homological spectral sequences associated with these two filtrations are isomorphic in many positions.

1. Introduction

An action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally modeled by the standard representation of T^n on \mathbb{C}^n . The orbit space of a local chart is isomorphic to a nonnegative cone $\mathbb{C}^n/T^n \cong \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$, thus the orbit space $Q = M/T^n$ of the whole manifold has a natural structure of manifold with corners. Points from interiors of k -dimensional faces of Q are the k -dimensional orbits of the action. For any face G of Q consider the stabilizer subgroup $T_G \subset T^n$ of points in the interior of G . The mapping sending the face G to the toric subgroup T_G is called characteristic data.

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For any manifold M with locally standard torus action having the orbit space Q there exists a principal T^n -bundle $Y \rightarrow Q$ such that M is equivariantly homeomorphic to the identification space $X = Y / \sim$, where \sim identifies points over a face $G \subset Q$ differing by the action of T_G [13]. Thus any manifold with locally standard action is uniquely determined, up to equivariant homeomorphism, by three objects: a manifold with corners Q , a principal torus bundle Y over Q (these bundles are encoded by their Euler classes lying in $H^2(Q; \mathbb{Z}^n)$), and characteristic data.

For a manifold with corners Q consider the dual poset S_Q . The elements of S_Q are the faces of Q and the order is given by reversed inclusion. If Q is the orbit space of a manifold with locally standard action, then S_Q is a simplicial poset (see Definition 2.1).

The description of topology of X in terms of the combinatorial data is difficult and, in general, far from being accomplished. The cohomology and equivariant cohomology rings are unknown and even Betti numbers haven't been explicitly calculated yet.

Nevertheless, there are several important particular cases which are known and well studied. If the orbit space Q is isomorphic to a simple polytope, the manifold X is called quasitoric. This particular case was introduced and studied in the seminal work of Davis and Januszkiewicz [6] and underlied the development of toric topology. Quasitoric manifolds are natural topological generalizations of smooth projective toric varieties. The reason which makes quasitoric manifolds feasible from topological viewpoint is that the orbit space has trivial topology (the convexity happens to be not so important).

This setting may be generalized to the case when all faces of Q are acyclic. This situation is very close to toric varieties or quasitoric manifolds and the answer is also very similar [8]:

$$H_{T^n}^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q]; \quad H^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q] / (\theta_1, \dots, \theta_n),$$

where $\mathbb{Z}[S_Q]$ is the face ring of the simplicial poset S_Q and $(\theta_1, \dots, \theta_n)$ is a regular sequence of degree 2 in $\mathbb{Z}[S_Q]$, determined by the characteristic data.

There are several papers where the calculation of topological invariants was performed for more general examples. In [1] we proved that whenever all proper faces of Q are acyclic and $Y \rightarrow Q$ is a trivial bundle, the equivariant cohomology ring is represented as a direct sum (as rings and as modules over $H^*(BT^n; \mathbb{Z})$):

$$H_{T^n}^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q] \oplus H^*(Q; \mathbb{Z}).$$

We also calculated Betti numbers and partly described the ring structure of $H^*(X, \mathbb{Z})$ when X is an orientable toric origami manifold with acyclic proper faces of the orbit space. This is a very restricted class of manifolds with locally standard actions, but even in this case many interesting phenomena sprang up. Betti numbers of 4-dimensional toric origami manifolds without any restrictions on proper faces were

calculated in [7]. The cohomology rings of 4-dimensional manifolds whose orbit spaces are polygons with holes were described in [12].

In [13] Yoshida introduced the cohomological spectral sequence converging to $H^*(X; \mathbb{Z})$ for any X , but generally this spectral sequence does not quickly collapse, so it is difficult to extract any explicit information, such as Betti numbers, from it. This approach requires an extra effort to obtain a concrete result. However, for some particular choices of X this extra effort can be done.

In this paper we study the homological structure of a manifold X , and related objects, using the filtration of X by orbit types

$$(1.1) \quad X_0 \subset X_1 \subset \dots \subset X_n = X.$$

Here X_i is the union of all T^n -orbits of dimension at most i , so $\dim X_i = 2i$. This filtration induces a spectral sequence $(E_X)_{p,q}^r \Rightarrow H_{p+q}(X)$, where $(E_X)_{p,q}^1 \cong H_{p+q}(X_p, X_{p-1})$. By dimensional reasons, $(E_X)_{p,q}^r = 0$ for $p < q$ and $r \geq 1$.

There is a natural topological filtration of Y which covers the orbit type filtration of X , and the map $f: Y \rightarrow X$ induces the map of homological spectral sequences

$$(1.2) \quad f_*^r: (E_Y)_{p,q}^r \rightarrow (E_X)_{p,q}^r.$$

If the proper faces of Q are acyclic, we prove that the map f_*^2 is an isomorphism for $p > q$. Thus every entry of $(E_X)_{p,q}^r$ away from the diagonal is known, at least if the structure of $(E_Y)_{p,q}^2$ is known.

To prove the above-mentioned isomorphism (Theorem 5.2), we place the maps $f_*^2: (E_Y)_{p,q}^2 \rightarrow (E_X)_{p,q}^2$ into a long exact sequence and show that certain intermediate terms of this sequence vanish. These intermediate terms are the cohomology modules $H^*(S_Q; \mathcal{I})$ of a graded sheaf \mathcal{I} on S_Q , whose values are the ideals in the homology algebra $H_*(T^n)$ generated by the vector subspaces $H_1(T_G) \subset H_1(T^n)$. The vanishing of these sheaf cohomology in certain degrees is the most nontrivial and essential part of the work. It follows from the duality:

$$(1.3) \quad H^{n-1-i}(S_Q; \mathcal{I}) \cong H_i(S_Q; \hat{\Pi}),$$

(Theorem 3.5) which holds for the homology manifold S_Q and extends the Poincare duality $H^{n-1-i}(S_Q; \mathbb{k}) \cong H_i(S_Q; \mathbb{k})$. Here $\hat{\Pi}$ is a cellular cosheaf on S_Q whose value on a face $G \subset Q$ is the ideal in $H_*(T^n)$ generated by the volume form of the submodule $H_*(T_G) \subset H_*(T^n)$.

We study this duality in a broader and quite natural setting. For a simplicial poset S there exists a Zeeman–McCrory spectral sequence $(E_{ZM})_{p,q}^r$. It converges to the homology of S , and its second page is the cohomology of local homology stacks \mathcal{U}_* on S . If S is a manifold, this sequence collapses at a second page and gives a standard proof of the Poincare duality. Thus Zeeman–McCrory spectral sequence can be roughly considered as a generalization of Poincare duality to non-manifolds.

We prove that there is a spectral sequence, starting with $H^*(S; \mathcal{U}_* \otimes \mathcal{I})$ and converging to $H_*(S; \widehat{\Pi})$ (Theorem 3.4). For homology manifolds it collapses to the isomorphism (1.3).

The work is organized as follows. In Section 2 we review the basic notions: simplicial posets, sheaves, cosheaves, and Zeeman–McCrory spectral sequence. The word “sheaf” will always mean “a sheaf over a finite poset”. It is not used in its broadest topological sense, but rather replaces the term stack or local coefficient system. In Section 3 we introduce the notion of homological characteristic function, define two objects associated with this object: the sheaf \mathcal{I} , and the cosheaf $\widehat{\Pi}$, and formulate Theorems 3.4 and 3.5, proving the duality (1.3). Theorem 3.4 is proved in Section 4, and Theorem 3.5 follows as its particular case. Preliminaries on manifolds with locally standard actions are given in Section 5. We introduce topological filtrations on Q , X , and Y , and formulate Theorem 5.2, which states that modules $(E_X)_{p,q}^r$ are isomorphic to $(E_Y)_{p,q}^2$ for $p > q$. Section 6 is devoted to the proof of Theorem 5.2; there we explain the connection of manifolds with torus actions and the sheaf-theoretical part of the work.

2. Sheaves and cosheaves over simplicial posets

2.1. Preliminaries on simplicial posets.

DEFINITION 2.1. *A finite partially ordered set (poset) is called simplicial if there exists a minimal element $\hat{0} \in S$ and, for any $I \in S$, the lower order ideal $\{J \in S \mid J \leq I\}$ is isomorphic to the boolean lattice $2^{[k]}$ (the poset of faces of a $(k-1)$ -dimensional simplex) for some $k \geq 0$.*

The elements of S are called *simplices*. The number k in the definition is denoted by $|I|$ and called the rank of a simplex I . Also set $\dim I = |I| - 1$. A simplex of rank 1 is called a *vertex*; the set of all vertices is denoted by $\text{Vert}(S)$. A subset $L \subset S$ closed under taking sub-simplices is called a simplicial subposet.

The notation $I \overset{i}{<} J$ is used whenever $I \leq J$ and $|J| - |I| = i$. If S is a simplicial poset, then for each $I \overset{2}{<} J \in S$, there exist exactly two simplices $J' \neq J''$ between I and J :

$$(2.1) \quad I \overset{1}{<} J', J'' \overset{1}{<} J.$$

For simplicial poset S a “sign convention” can be chosen. It means that we can associate an incidence number $[J : I] = \pm 1$ with any pair $I \overset{1}{<} J \in S$ such that

$$(2.2) \quad [J : J'] \cdot [J' : I] + [J : J''] \cdot [J'' : I] = 0$$

for any combination (2.1). The choice of sign convention is the same as orienting each simplex in S . We fix an arbitrary sign convention and use it in the following considerations.

Notice that the set of simplices of any finite simplicial complex obviously forms a simplicial poset. Thus the notion of simplicial poset is a straightforward generalization of abstract simplicial complex.

For $I \in S$ consider the following subset of S :

$$\text{st}_S^\circ I = \{J \in S \mid J \geq I\},$$

called the *open star* of I . It is easily seen that $S \setminus \text{st}_S^\circ I$ is a simplicial subposet of S .

We also define the *link* of a simplex $I \in S$:

$$\text{lk}_S I = \{J \in S \mid J \geq I\}.$$

This set inherits the order relation from S , and $\text{lk}_S I$ is a simplicial poset with respect to this order, with the minimal element I . The reason why we used to different notation for the same thing is that it is convenient to distinguish between $\text{st}_S^\circ I$, which is considered as a subset of S (but not a subposet!), and $\text{lk}_S I$, which is considered as a simplicial poset on its own (and which is, in general, not included in S as a subposet in any meaningful way). Note that $\text{lk}_S \hat{0} = S$.

Let S' be the barycentric subdivision of S . By definition, S' is a simplicial complex on the set $S \setminus \hat{0}$ whose simplices are the chains of elements of S . By definition, the geometric realization of S is the geometric realization of its barycentric subdivision $|S| \stackrel{\text{def}}{=} |S'|$. One can also think of $|S|$ as a CW-complex with simplicial cells. Such topological models of simplicial posets were called *simplicial cell complexes* and were studied in [3].

A poset S is called *pure* if all its maximal elements have equal dimensions. A poset S is pure whenever S' is pure.

In the following \mathbb{k} denotes the ground ring; it may be either a field or the ring of integers. The (co)homology of simplicial poset S mean the (co)homology of its geometrical realization $|S|$. If the coefficient ring in the notation of (co)homology is omitted, it is supposed to be \mathbb{k} .

DEFINITION 2.2. *Simplicial complex K of dimension $n - 1$ is called Buchsbaum (over \mathbb{k}) if $\tilde{H}_i(\text{lk}_K I; \mathbb{k}) = 0$ for all $\hat{0} \neq I \in K$ and $i \neq n - 1 - |I|$. If K is Buchsbaum and, moreover, $\tilde{H}_i(K; \mathbb{k}) = 0$ for $i \neq n - 1$ then K is called Cohen–Macaulay.*

Simplicial poset S is called Buchsbaum (resp. Cohen–Macaulay) if S' is a Buchsbaum (resp. Cohen–Macaulay) simplicial complex.

REMARK 2.3. By [11, Sec.6], S is Buchsbaum whenever $\tilde{H}_i(\text{lk}_S I; \mathbb{k}) = 0$ for all $\hat{0} \neq I \in S$ and $i \neq n - 1 - |I|$. Similarly, S is Cohen–Macaulay whenever $\tilde{H}_i(\text{lk}_S I; \mathbb{k}) = 0$ for all $I \in S$ and $i \neq n - 1 - |I|$.

Typical examples of Buchsbaum posets are triangulations (and, more generally, simplicial cell decompositions) of manifolds. Typical examples of Cohen–Macaulay posets are triangulations of spheres. A poset S is Buchsbaum whenever all its proper links are Cohen–Macaulay.

One can easily check that whenever S is Buchsbaum and connected, then S is pure. In the following only pure simplicial posets are considered.

2.2. Cellular sheaves. Let $\text{MOD}_{\mathbb{k}}$ be the category of \mathbb{k} -modules. The notation $\dim V$ is used for the rank of a \mathbb{k} -module V .

Each simplicial poset S determines a small category $\text{CAT}(S)$ whose objects are the elements of S and the morphisms are the inequalities $I \leq J$. A *cellular sheaf* [5] (or a stack [10], or a local coefficient system elsewhere) on S is a covariant functor $\mathcal{A}: \text{CAT}(S) \rightarrow \text{MOD}_{\mathbb{k}}$. We simply call \mathcal{A} a sheaf on S and hope this will not lead to a confusion, since other meanings of this word do not appear in the paper. The maps $\mathcal{A}(J_1 \leq J_2)$ are called *restriction maps*. The cochain complex $(C^*(S; \mathcal{A}), d)$ is defined as follows:

$$C^*(S; \mathcal{A}) = \bigoplus_{i \geq -1} C^i(S; \mathcal{A}), \quad C^i(S; \mathcal{A}) = \bigoplus_{\dim I = i} \mathcal{A}(I),$$

$$d: C^i(S; \mathcal{A}) \rightarrow C^{i+1}(S; \mathcal{A}), \quad d = \bigoplus_{I \leq I', \dim I = i} [I' : I] \mathcal{A}(I \leq I').$$

The sign convention (2.2) implies that $d^2 = 0$. Thus $(C^*(S; \mathcal{A}), d)$ is a differential complex. Define the cohomology of \mathcal{A} as the cohomology of this complex:

$$(2.3) \quad H^*(S; \mathcal{A}) \stackrel{\text{def}}{=} H^*(C^*(S; \mathcal{A}), d).$$

REMARK 2.4. Cohomology of \mathcal{A} defined this way coincide with any other meaningful definition of cohomology. For example the derived functors of the functor of global sections give the same groups as (2.3) (refer to [5] for a broad exposition of this subject).

A sheaf \mathcal{A} on S can be restricted to a simplicial subposet $L \subset S$. The complexes $(C^*(L, \mathcal{A}), d)$ and $(C^*(S; \mathcal{A})/C^*(L; \mathcal{A}), d)$ are defined as usual. The latter complex gives rise to a relative version of sheaf cohomology: $H^*(S, L; \mathcal{A})$.

REMARK 2.5. It is standard in topological literature to consider cellular sheaves which do not take values on $\hat{0} \in S$, since in general this element does not have a geometrical meaning. However, this extra value $\mathcal{A}(\hat{0})$ will be important in the considerations of Section 6. Therefore the cohomology group may be nontrivial in degree $-1 = \dim \hat{0}$. If a sheaf \mathcal{A} is defined on S , then we can consider its truncated version $\underline{\mathcal{A}}$ which coincides with \mathcal{A} on $S \setminus \{\hat{0}\}$ and vanishes on $\hat{0}$.

The notions of maps, (co)kernels, (co)images, tensor products of sheaves over S are defined in an obvious componentwise manner. For example, if \mathcal{A} and \mathcal{B} are two sheaves on S , then $\mathcal{A} \otimes \mathcal{B}$ is a sheaf on S with values $(\mathcal{A} \otimes \mathcal{B})(I) = \mathcal{A}(I) \otimes \mathcal{B}(I)$ and restriction maps $(\mathcal{A} \otimes \mathcal{B})(I \leq J) = \mathcal{A}(I \leq J) \otimes \mathcal{B}(I \leq J)$. In the realm of finite simplicial posets the distinction between “sheaves” and “presheaves” vanishes, which makes things simpler than they are in algebraic geometry.

EXAMPLE 2.6. Let W be a \mathbb{k} -module. By abuse of notation let W denote the globally constant sheaf on S . It takes the constant value W on $I \neq \hat{0}$ and vanishes on $\hat{0}$. All nontrivial restriction maps are identity isomorphisms. If W is torsion-free, we have $H^*(S; W) \cong H^*(S; \mathbb{k}) \otimes W$ by the universal coefficients formula.

EXAMPLE 2.7. A locally constant sheaf valued by $W \in \text{MOD}_{\mathbb{k}}$ is a sheaf \mathcal{W} which satisfies $\mathcal{W}(\hat{0}) = 0$, $\mathcal{W}(I) \cong W$ for $I \neq \hat{0}$ and all nontrivial restriction maps are isomorphisms (but may be not identity isomorphisms).

EXAMPLE 2.8. Following [10], define i -th local homology sheaf \mathcal{U}_i on S by setting $\mathcal{U}_i(\hat{0}) = 0$ and

$$(2.4) \quad \mathcal{U}_i(J) = H_i(S, S \setminus \text{st}_S^\circ J; \mathbb{k})$$

for $J \neq \hat{0}$. The restriction maps $\mathcal{U}_i(J_1 < J_2)$ are induced by inclusions of subsets $\text{st}_S^\circ J_2 \hookrightarrow \text{st}_S^\circ J_1$. Standard topological arguments imply that a simplicial poset S is Buchsbaum if and only if $\mathcal{U}_i = 0$ for $i \neq n - 1$ (see also Remark 2.16 below).

DEFINITION 2.9. *Buchsbaum simplicial poset S is called homology manifold (orientable over \mathbb{k}) if its local homology sheaf \mathcal{U}_{n-1} is isomorphic to the constant sheaf \mathbb{k} .*

S is an orientable homology manifold if and only if its geometrical realization is an orientable homology manifold in a usual topological sense.

2.3. Cosheaves. A cellular cosheaf [5] is a contravariant functor $\hat{\mathcal{A}}: \text{CAT}(S)^{op} \rightarrow \text{MOD}_{\mathbb{k}}$. The homology of a cosheaf are defined similar to the cohomology of a sheaf:

$$\begin{aligned} C_*(S; \hat{\mathcal{A}}) &= \bigoplus_{i \geq -1} C_i(S; \hat{\mathcal{A}}) & C_i(S; \hat{\mathcal{A}}) &= \bigoplus_{\dim I = i} \mathcal{A}(I) \\ d: C_i(S; \hat{\mathcal{A}}) &\rightarrow C_{i-1}(S; \hat{\mathcal{A}}), & d &= \bigoplus_{I >_1 I', \dim I = i} [I : I'] \hat{\mathcal{A}}(I \geq I'), \\ H_*(S; \hat{\mathcal{A}}) &\stackrel{\text{def}}{=} H_*(C_*(S; \hat{\mathcal{A}}), d). \end{aligned}$$

EXAMPLE 2.10. Each locally constant sheaf \mathcal{W} on S determines the locally constant cosheaf $\widehat{\mathcal{W}}$ by inverting all maps, i.e. $\widehat{\mathcal{W}}(I) \cong \mathcal{W}(I)$ and $\widehat{\mathcal{W}}(I > J) = (\mathcal{W}(J < I))^{-1}$.

REMARK 2.11. Notice that the notation $H_*(S; \mathbb{k})$ can mean either the homology of the geometric realization $|S|$ or the homology of a globally constant cosheaf \mathbb{k} on S . Obviously these two meanings are consistent, and the same for cohomology of a constant sheaf.

2.4. Coskeleton filtration and dual faces. In the following we suppose that S is pure and $\dim S = n - 1$.

CONSTRUCTION 2.12. Let us recall the construction of *coskeleton filtration* on $|S|$. Consider the barycentric subdivision S' of the pure simplicial poset S . By

definition, S' is a simplicial complex on the set $S \setminus \hat{0}$ and k -simplices of S' have the form $(I_0 < I_1 < \dots < I_k)$, where $I_i \in S \setminus \hat{0}$. For each $I \in S \setminus \{\hat{0}\}$ consider the subcomplex of the barycentric subdivision:

$$G_I = \{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 \geq I\} \subset S',$$

and the subsets

$$\partial G_I = \{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 > I\} \subset S', \text{ and } G_I^\circ = G_I \setminus \partial G_I.$$

It is easily seen that $\dim G_I = n - 1 - \dim I$ since S is pure. We have $G_I \subset G_J$ whenever $J < I$. The complex G_I (or its geometrical realization $|G_I|$) is called the *face* or the *pseudocell* of $|S|$ dual to $I \in S$. The boundary ∂G_I of a face G_I is the union of some faces of smaller dimensions.

Let $S_i = \bigcup_{\dim G_I \leq i} G_I$ for $-1 \leq i \leq n-1$. Thus S_i is a simplicial subcomplex of S' . The filtration

$$(2.5) \quad \emptyset = S_{-1} \subset S_0 \subset S_1 \subset \dots \subset S_{n-1} = S',$$

and the corresponding topological filtration

$$(2.6) \quad \emptyset = |S_{-1}| \subset |S_0| \subset |S_1| \subset \dots \subset |S_{n-1}| = |S|,$$

are called the coskeleton filtrations of S' and $|S|$ respectively [10].

For a pair $I \stackrel{1}{<} J \in S$ consider the map:

$$(2.7) \quad m_{I,J}^q: H_{q+\dim G_I}(G_I, \partial G_I) \rightarrow H_{q+\dim G_{I-1}}(\partial G_I) \rightarrow \\ \rightarrow H_{q+\dim G_{I-1}}(\partial G_I, \partial G_I \setminus G_J^\circ) \cong H_{q+\dim G_J}(G_J, \partial G_J),$$

where the first map is the connecting homomorphism in the long exact sequence of homology for the pair $(G_I, \partial G_I)$, and the last isomorphism is due to excision. The homology spectral sequence associated with filtration (2.6) runs

$$(E_S)_{p,q}^1 = H_{p+q}(S_p, S_{p-1}) \Rightarrow H_{p+q}(S).$$

The first differential $(d_S)^1$ is the sum of the maps $m_{I,J}^q$ over all pairs $I \stackrel{1}{<} J$, $I, J \in S$.

CONSTRUCTION 2.13. Given a sign convention on S , for each q consider the sheaf \mathcal{H}_q on S given by

$$\mathcal{H}_q(I) = H_{q+\dim G_I}(G_I, \partial G_I)$$

for $I \neq \hat{0}$, and $\mathcal{H}_q(\hat{0}) = 0$. For neighboring simplices $I \stackrel{1}{<} J$ define the restriction map as $\mathcal{H}_q(I \stackrel{1}{<} J) = [J : I] m_{I,J}^q$. For general $I \stackrel{k}{<} J$ consider any saturated chain in S between I and J :

$$I \stackrel{1}{<} J_1 \stackrel{1}{<} \dots \stackrel{1}{<} J_{k-1} \stackrel{1}{<} J,$$

and set

$$\mathcal{H}_q(I < J) \stackrel{\text{def}}{=} \mathcal{H}_q(J_{k-1} < J) \circ \dots \circ \mathcal{H}_q(I < J_1).$$

LEMMA 2.14. *The map $\mathcal{H}_q(I < J)$ thus defined does not depend on a choice of saturated chain between I and J .*

PROOF. The differential $(d_S)^1$ satisfies $((d_S)^1)^2 = 0$, thus $m_{J',J}^q \circ m_{I,J'}^q + m_{J'',J}^q \circ m_{I,J''}^q = 0$. By combining this with (2.2) we see that $\mathcal{H}_q(I < J)$ is independent of the chain if its length is 2. In general, since $\{T \mid I \leq T \leq J\}$ is a boolean lattice, any two saturated chains between I and J are connected by a sequence of elementary flips $[J_k \stackrel{1}{<} T_1 \stackrel{1}{<} J_{k+2}] \rightsquigarrow [J_k \stackrel{1}{<} T_2 \stackrel{1}{<} J_{k+2}]$ and the statement follows. \square

Thus the sheaves \mathcal{H}_q are well defined. They will be called the structure sheaves of S . From the definition of a cochain complex directly follows

COROLLARY 2.15. *The cochain complexes of structure sheaves coincide with $(E_S)_{*,*}^1$ up to change of indices:*

$$((E_S)_{*,q}^1, (d_S)^1) \cong (C^{n-1-*}(\mathcal{H}_q), d).$$

REMARK 2.16. There exists an isomorphism of sheaves

$$(2.8) \quad \mathcal{H}_q \cong \mathcal{U}_{q+n-1},$$

where \mathcal{U}_* are the sheaves of local homology defined in Example 2.8. Indeed, it can be shown that $H_i(S, S \setminus \text{st}_S^\circ I) \cong H_{i-\dim I}(G_I, \partial G_I)$ and these isomorphisms can be chosen compatible with restriction maps. For simplicial complexes this fact is proved in [10, Sec.6.1]; the case of simplicial posets is rather similar. Note that the definition of \mathcal{H}_* depends on the sign convention while \mathcal{U}_* does not. This makes no contradiction since the isomorphism (2.8) itself depends on the choice of orientations.

The isomorphism (2.8) implies that S is Buchsbaum if and only if $\mathcal{H}_q = 0$ for $q \neq 0$. Simplicial poset S is an orientable manifold if it is Buchsbaum and, moreover, $\mathcal{H}_0 \cong \mathbb{k}$.

2.5. Zeeman–McCrory spectral sequence. From the considerations of the previous subsection easily follows

STATEMENT 2.17 (McCrory, [10]). *There exists a spectral sequence, located in fourth quadrant,*

$$(2.9) \quad (E_{ZM})_{p,q}^r, \quad d^r : (E_{ZM})_{p,q}^r \rightarrow (E_{ZM})_{p-r,q+r-1}^r;$$

$$(2.10) \quad (E_{ZM})_{p,q}^2 \cong H^{n-1-p}(S; \mathcal{U}_{n-1+q}) \Rightarrow H_{p+q}(S; \mathbb{k}).$$

It is isomorphic to the homological spectral sequence, associated with the coskeleton filtration of $|S|$.

For us, however, it will be more convenient to work with structure sheaves \mathcal{H}_* rather than local homology sheaves \mathcal{U}_* . For a Buchsbaum simplicial poset the sheaf \mathcal{H}_i vanish for $i \neq 0$. Thus $(E_{ZM})_{p,q}^2 = 0$ for $q \neq 0$ and the spectral sequence collapses at the second page inducing the isomorphism

$$H^{n-1-p}(S; \mathcal{H}_0) \cong H_p(S; \mathbb{k}).$$

When S is an orientable homology manifold, this gives a Poincare duality isomorphism

$$H^{n-1-p}(S; \mathbb{k}) \cong H_p(S; \mathbb{k}).$$

2.6. Corefinements of sheaves. In this section we develop a technical notion which will be used further in the proofs.

Let \mathcal{A} be a sheaf on S . Define a cosheaf $\hat{\mathcal{A}}'$ on the barycentric subdivision S' by

$$\hat{\mathcal{A}}'(I_1 < \dots < I_k) = \mathcal{A}(I_1)$$

with corestriction maps determined naturally by restriction maps of \mathcal{A} :

$$\hat{\mathcal{A}}'((I_1 < \dots < I_k) \supset (J_1 < \dots < J_s)) = \mathcal{A}(I_1 \leq J_1).$$

We call $\hat{\mathcal{A}}'$ a corefinement of a sheaf \mathcal{A} . The faces G_I and their boundaries ∂G_I are the simplicial subcomplexes of S' so one can restrict $\hat{\mathcal{A}}'$ to them. Next lemma follows easily from the definitions.

LEMMA 2.18.

$$H_q(S_p, S_{p-1}, \hat{\mathcal{A}}') \cong \bigoplus_{I, \dim G_I = p} H_q(G_I, \partial G_I; \hat{\mathcal{A}}').$$

Similar to (2.7) there is a map

$$(2.11) \quad m_{I,J}^{q,\mathcal{A}}: H_{q+\dim G_I}(G_I, \partial G_I; \hat{\mathcal{A}}') \rightarrow H_{q+\dim G_{I-1}}(\partial G_I; \hat{\mathcal{A}}') \rightarrow \\ \rightarrow H_{q+\dim G_{I-1}}(\partial G_I, \partial G_I \setminus G_J^\circ; \hat{\mathcal{A}}') \cong H_{q+\dim G_J}(G_J, \partial G_J; \hat{\mathcal{A}}').$$

These maps allow to define new sheaves $\overline{\mathcal{A}}_q$ on S by setting $\overline{\mathcal{A}}_q(I) = H_{q+\dim G_I}(G_I, \partial G_I; \hat{\mathcal{A}}')$ with restriction maps defined similar to Construction 2.13.

LEMMA 2.19. *If $\mathcal{A}(I)$ is torsion-free for all $I \in S$, then there exist natural isomorphisms*

$$H_r(G_I, \partial G_I; \hat{\mathcal{A}}') \cong H_r(G_I, \partial G_I; \mathbb{k}) \otimes \mathcal{A}(I).$$

The maps $m_{I,J}^{q,\mathcal{A}}$ coincide with $m_{I,J}^q \otimes \mathcal{A}(I < J)$ up to these isomorphisms. Thus the sheaf $\overline{\mathcal{A}}_q$ is isomorphic to $\mathcal{H}_q \otimes \mathcal{A}$.

PROOF. By the definition of $\hat{\mathcal{A}}'$ we have

$$H_r(G_I, \partial G_I; \hat{\mathcal{A}}') \cong H_r(G_I, \partial G_I; \mathcal{A}(I)),$$

since the value of $\hat{\mathcal{A}}'$ on all simplices of G_I° is exactly $\mathcal{A}(I)$. The rest follows from universal coefficients formula. \square

3. Exterior algebras and characteristic functions

Let V be a free \mathbb{k} -module of dimension N . Let $\Lambda[V]$ denote the free exterior algebra generated by V , that is the quotient of a free tensor algebra $T[V]$ by the relations $v \otimes v = 0$ for all $v \in V$. The algebra $\Lambda[V]$ is graded by degrees of exterior forms.

DEFINITION 3.1. *Let us fix a simplicial poset S and a locally constant sheaf \mathcal{V} on S . A collection of vectors $\{\omega_i \in \mathcal{V}(i) \mid i \in \text{Vert}(S)\}$ is called a homological \mathbb{k} -characteristic function if it satisfies the following $(*_\mathbb{k})$ -condition:*

For each simplex $I \in S \setminus \hat{0}$ whose vertices are i_1, \dots, i_k , the vectors

$$\mathcal{V}(i_1 \leq I)(\omega_{i_1}), \dots, \mathcal{V}(i_k \leq I)(\omega_{i_k}) \in \mathcal{V}(I)$$

are linearly independent over \mathbb{k} and span a direct summand in $\mathcal{V}(I)$.

For a locally constant sheaf \mathcal{V} on S , valued by the vector space V , consider the sheaf $\mathcal{L} = \Lambda[\mathcal{V}]$ of graded exterior algebras generated \mathcal{V} . This means that $\mathcal{L}(I) = \Lambda[\mathcal{V}(I)]$, and $\mathcal{L}(I \leq J)$ is an isomorphism of graded exterior algebras generated by the isomorphism $\mathcal{V}(I \leq J): \mathcal{V}(I) \rightarrow \mathcal{V}(J)$ in degree one levels. Let $\hat{\mathcal{L}}$ denote the locally constant cosheaf of exterior algebras corresponding to a sheaf \mathcal{L} (see Example 2.10).

Let $\{\omega_i \in \mathcal{V}(i) \mid i \in \text{Vert}(S)\}$ be a homological characteristic function. If i is a vertex of a simplex I , then the restriction map $\mathcal{V}(i \leq I)$ sends the vector $\omega_i \in \mathcal{V}(i)$ to some vector in $\mathcal{V}(I)$. By abuse of notation we denote the target vector by the same letter ω_i . So far the definition of homological characteristic function implies that the set $\{\omega_{i_1}, \dots, \omega_{i_k}\}$ freely spans a direct summand of $\mathcal{V}(I)$ whenever i_1, \dots, i_k are vertices of I . Note, that $\mathcal{L}(I)$ is an exterior algebra generated by $\mathcal{V}(I)$, so the vectors ω_i can be considered as linear forms in $\mathcal{L}(I)$.

CONSTRUCTION 3.2. Consider a subsheaf $\mathcal{I} \subset \mathcal{L}$, defined as follows. For a simplex I with vertices i_1, \dots, i_k we set the value of \mathcal{I} on I to be the ideal of $\mathcal{L}(I)$, generated by the linear forms:

$$\mathcal{I}(I) = (\omega_{i_1}, \dots, \omega_{i_k}).$$

It is easily seen that whenever $I \leq J$, the restriction map $\mathcal{L}(I \leq J)$ sends the ideal $\mathcal{I}(I)$ generated by the smaller set of elements into the ideal $\mathcal{I}(J)$ generated by the larger set of elements. Thus the restriction maps of the sheaf \mathcal{I} are induced from those of \mathcal{L} and are well defined.

CONSTRUCTION 3.3. Let us define another type of ideals associated with a characteristic function.

Let $J = \{i_1, \dots, i_k\}$ be a nonempty subset of vertices of a simplex $I \in S$. Consider the element $\pi_J \in \mathcal{L}(I) = \hat{\mathcal{L}}(I)$, $\pi_J = \bigwedge_{i \in J} \omega_i$. By the definition of characteristic function, the elements $\{\omega_i \mid i \in J\}$ are linearly independent, thus π_J is a non-zero form of degree $|J|$. Let $\Pi_J \subset \mathcal{L}(I)$ be the principal ideal generated by π_J . The

restriction maps $\mathcal{L}(I < I')$ (and corestriction maps $\widehat{\mathcal{L}}(I' > I) = \mathcal{L}(I < I')^{-1}$) identify $\Pi_J \subset \mathcal{L}(I)$ with $\Pi_J \subset \mathcal{L}(I')$.

Let us define a subcosheaf $\widehat{\Pi}$ of ideals in $\widehat{\mathcal{L}}$. If J is the whole set of vertices of a simplex $I \neq \hat{0}$ we define $\widehat{\Pi}(I) \stackrel{\text{def}}{=} \Pi_J \subset \widehat{\mathcal{L}}(I)$. If $I' < I$, the corestriction map $\widehat{\mathcal{L}}(I' > I)$ injects $\widehat{\Pi}(I')$ into $\widehat{\Pi}(I)$, since the form $\pi_{I'}$ is divisible by π_I . Thus $\widehat{\Pi}$ is a well-defined graded sub-cosheaf of $\widehat{\mathcal{L}}$. We formally set $\widehat{\Pi}(\hat{0}) = 0$.

Now we can formulate our main homological results.

THEOREM 3.4. *Let S be a pure simplicial poset of dimension $n-1$, and $\mathcal{I}, \widehat{\Pi}$ the sheaf and cosheaf over S , determined by some homological \mathbb{k} -characteristic function. Then there exists a spectral sequence*

$$E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{H}_k \otimes \mathcal{I}) \Rightarrow H_{s+k}(S; \widehat{\Pi}),$$

$$d^r : E_{s,k}^r \rightarrow E_{s-r, k+r-1}^r$$

which respects the inner gradings of \mathcal{I} and $\widehat{\Pi}$.

If S is Buchsbaum, the spectral sequence of Theorem 3.4 collapses at a second page and implies

THEOREM 3.5. *For Buchsbaum simplicial poset S of dimension $n-1$ there exists an isomorphism $H^k(S; \mathcal{H}_0 \otimes \mathcal{I}) \cong H_{n-1-k}(S; \widehat{\Pi})$ which respects the inner gradings of \mathcal{I} and $\widehat{\Pi}$.*

COROLLARY 3.6. *If S is a homology $(n-1)$ -manifold, then there is an isomorphism $H^k(S; \mathcal{I}) \cong H_{n-1-k}(S; \widehat{\Pi})$, respecting the inner gradings.*

Let $\mathcal{I}^{(q)}, \widehat{\Pi}^{(q)}$ denote the homogeneous parts of inner degree q of the corresponding sheaves $\mathcal{I}, \widehat{\Pi}$.

COROLLARY 3.7 (Key corollary). *If S is a Buchsbaum simplicial poset, then $H^j(S; \mathcal{H}_0 \otimes \mathcal{I}^{(q)}) = 0$ for $j \leq n-1-q$.*

PROOF. By Theorem 3.5, it is sufficient to prove that $H_j(S; \widehat{\Pi}^{(q)}) = 0$ for $j \geq q$. The ideal $\widehat{\Pi}(I) = \Pi_I$ is generated by the element π_I of degree $|I| = \dim I + 1$. Thus $\Pi_I^{(q)} = 0$ for $q \leq \dim I$. Hence the corresponding part of the chain complex vanishes, and the homology in these degrees vanish as well. \square

REMARK 3.8. The exterior forms of the top power, $\Lambda[V]^{(N)}$, lie in every ideal $\mathcal{I}(I)$ and $\widehat{\Pi}(I)$. Thus the isomorphism of Theorem 3.5, when restricted to the top degree, gives the Poincare duality:

$$H^k(S; \mathcal{H}_0) = H^k(S; \mathcal{H}_0 \otimes \mathcal{I}^{(N)}) \cong H_{n-1-k}(S; \widehat{\Pi}^{(N)}) = H_{n-1-k}(S; \mathbb{k}).$$

The restriction of the spectral sequence of Theorem 3.4 to the top degree gives the Zeeman–McCrory spectral sequence in a similar way.

4. Proof of Theorem 3.4

The idea of proof is the following. We construct a filtered double differential complex $\mathcal{X}_{k,l}$ and then play with various spectral sequences converging to its total homology.

Before we proceed we need a small technical lemma. Let $J \in S$. If i is a vertex of J , we have a map $\eta_i: \Pi_i \hookrightarrow \mathcal{I}(J)$, which includes the ideal Π_i generated by a linear form ω_i into the ideal $\mathcal{I}(J)$ generated by a larger set of linear forms.

Consider the sequence of maps

$$(4.1) \quad 0 \leftarrow \mathcal{I}(J) \xleftarrow{\eta} \bigoplus_{\substack{I, \dim I=0 \\ I \subseteq J}} \Pi_I \xleftarrow{\xi} \bigoplus_{\substack{I, \dim I=1 \\ I \subseteq J}} \Pi_I \xleftarrow{\xi} \bigoplus_{\substack{I, \dim I=2 \\ I \subseteq J}} \Pi_I \xleftarrow{\xi} \dots$$

where η is the direct sum of the maps η_i over $i \in \text{Vert}(S)$, $i \leq J$; and ξ is the direct sum of inclusion maps $\Pi_I \hookrightarrow \Pi_{I'}$, each rectified by the incidence sign $[I : I']$. The sign convention obviously implies that (4.1) is a differential complex. But what is more important,

LEMMA 4.1. *The sequence (4.1) is exact.*

PROOF. This is very similar to the Taylor resolution of monomial ideal in commutative polynomial ring (or Koszul resolution), but our situation is a bit different, since Π_I are not free modules over Λ . Anyway, the proof is similar to commutative case: exactness of (4.1) follows from inclusion-exclusion principle. To make things precise (and also to tackle the case $\mathbb{k} = \mathbb{Z}$) we proceed as follows.

By $(*_\mathbb{k})$ -condition, the subspace $\langle \omega_j \mid j \in J \rangle$ is a direct summand in $V \cong \mathbb{k}^N$. Let $\{\nu_1, \dots, \nu_N\}$ be a basis of V such that its first $|J|$ vectors are exactly ω_j , $j \in J$. We simply identify J with the subset $\{1, \dots, |J|\} \subseteq [N]$ by abuse of notation. The module $\Lambda[V]$ splits in multidegree components: $\Lambda = \bigoplus_{A \subseteq [N]} \Lambda_A$, where Λ_A is a 1-dimensional \mathbb{k} -module generated by $\bigwedge_{i \in A} \nu_i$. All modules and maps in (4.1) respect this splitting. Thus (4.1) can be written as

$$0 \longleftarrow \bigoplus_{A \cap J \neq \emptyset} \Lambda_A \longleftarrow \bigoplus_{I \subseteq J, |I|=1} \bigoplus_{A \supseteq I} \Lambda_A \longleftarrow \bigoplus_{I \subseteq J, |I|=2} \bigoplus_{A \supseteq I} \Lambda_A \longleftarrow \dots,$$

$$\bigoplus_{A, A \cap J \neq \emptyset} \left(0 \longleftarrow \Lambda_A \longleftarrow \bigoplus_{I \subseteq A \cap J, |I|=1} \Lambda_A \longleftarrow \bigoplus_{I \subseteq A \cap J, |I|=2} \Lambda_A \longleftarrow \dots \right).$$

For each A , the homology of the complex in brackets coincides with $\tilde{H}_*(\Delta_{A \cap J}; \Lambda_A) \cong \tilde{H}_*(\Delta_{A \cap J}; \mathbb{k})$, the reduced simplicial homology of the simplex on the set $A \cap J \neq \emptyset$. Thus homology vanishes. \square

Let us define a cosheaf \mathcal{N} on S taking values in graded differential complexes. We set $\mathcal{N}(I) = C_*(G_I; \Pi_I)$, the simplicial chains of the simplicial complex G_I . The corestriction maps $\mathcal{N}(I > J)$ are naturally induced by inclusions of faces $G_I \hookrightarrow G_J$ and inclusions of coefficient modules $\hat{\Pi}(I > J): \Pi_I \hookrightarrow \Pi_J$.

The chain complex

$$\mathcal{X}_{*,*} = (C_*(S; \mathcal{N}); d_H), \quad \mathcal{X}_{k,l} = \bigoplus_{I, \dim I=k} C_l(G_I; \Pi_I)$$

is a double complex. It has the horizontal homological differential $d_H: \mathcal{X}_{k,l} \rightarrow \mathcal{X}_{k-1,l}$ (sheaf-differential) and the vertical differential $d_V: C_l(G_I; \Pi_I) \rightarrow C_{l-1}(G_I; \Pi_I)$ (inner differential). The differentials commute, $d_H d_V = d_V d_H$, so we can form a totalized differential complex

$$\mathcal{X}_j = \bigoplus_{k+l=j} \mathcal{X}_{k,l}, \quad d_{Tot} = d_H + (-1)^k d_V: \mathcal{X}_j \rightarrow \mathcal{X}_{j-1}.$$

LEMMA 4.2. $H_k(\mathcal{X}, d_{Tot}) \cong H_k(S; \hat{\Pi})$.

PROOF. Consider the vertical spectral sequence [9] converging to $H_k(\mathcal{X}, d_{Tot})$:

$$(E_V)_{*,*}^r, \quad (d_V)_r: (E_V)_{k,l}^r \rightarrow (E_V)_{k-r,l+r-1}^r,$$

which at first computes vertical homology, then horizontal. We have

$$(E_V)_{k,l}^1 = \bigoplus_{I, \dim I=k} H_l(G_I; \Pi_I).$$

Since G_I is contractible, $H_l(G_I; \Pi_I) = 0$ for $l \neq 0$ and $H_0(G_I; \Pi_I) = \Pi_I$. Thus

$$(E_V)_{k,l}^1 = \begin{cases} \bigoplus_{\dim I=k} \Pi_I = C_k(S; \hat{\Pi}), & \text{if } l = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$(E_V)_{k,l}^2 = \begin{cases} H_k(S; \hat{\Pi}), & \text{if } l = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The spectral sequence collapses at the second page, thus $H_k(\mathcal{X}, d_{Tot}) \cong H_k(S; \hat{\Pi})$. \square

Our next goal is to compute the homology of totalization by first computing the horizontal homology, then vertical. Recall that G_I is a simplicial subcomplex of S' , so the module $C_*(G_I; \Pi_I)$ is considered as the chain complex of the constant cosheaf Π_I . Let the cosheaf $\hat{\mathcal{I}}'$ be the corefinement of the sheaf \mathcal{I} (recall this notion from subsection 2.6).

LEMMA 4.3. *The sequence*

$$(4.2) \quad 0 \longleftarrow C_*(S'; \hat{\mathcal{I}}') \longleftarrow \bigoplus_{I, \dim I=0} C_*(G_I; \Pi_I) \longleftarrow \bigoplus_{I, \dim I=1} C_*(G_I; \Pi_I) \longleftarrow \dots$$

is exact.

PROOF. Since all the maps $C_*(G_I; \Pi_I) \rightarrow C_*(G_I; \Pi_I)$ are induced by inclusions of simplicial subcomplexes, the sequence (4.2) decomposes as the direct sum over all simplices $\Delta = (I_1 < \dots < I_k) \in S'$:

$$\bigoplus_{\Delta \in S'} \left(0 \longleftarrow \widehat{\mathcal{I}}'(\Delta) \longleftarrow \bigoplus_{\substack{I, \dim I=0 \\ \Delta \in G_I}} \Pi_I \longleftarrow \bigoplus_{\substack{I, \dim I=1 \\ \Delta \in G_I}} \Pi_I \longleftarrow \dots \right)$$

Since the condition $\Delta \in G_I$ is equivalent to $I_1 \geq I$, and by the definition of corefinement $\widehat{\mathcal{I}}'$, the expression in brackets is equal to

$$0 \longleftarrow \mathcal{I}(I_1) \longleftarrow \bigoplus_{\substack{I, \dim I=0 \\ I \leq I_1}} \Pi_I \longleftarrow \bigoplus_{\substack{I, \dim I=1 \\ I \leq I_1}} \Pi_I \longleftarrow \dots$$

This sequence is exact by Lemma 4.1 □

Let us return to the double complex \mathcal{X} and consider its horizontal spectral sequence

$$(E_H)_{*,*}^r, \quad (d_H)_r: (E_H)_{k,l}^r \rightarrow (E_H)_{k+r-1,l-r}^r$$

which computes horizontal homology first, then vertical.

LEMMA 4.4. $H_l(\mathcal{X}, d_{Tot}) \cong H_l(S'; \widehat{\mathcal{I}}')$.

PROOF. By Lemma 4.3 the horizontal homology of \mathcal{X} vanishes except in degree $k = 0$, where it is isomorphic to $C_*(S'; \widehat{\mathcal{I}}')$. Thus

$$(E_H)_{k,l}^2 \cong \begin{cases} H_l(S'; \widehat{\mathcal{I}}'), & \text{if } k = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The spectral sequence collapses and the statement follows. □

Finally, we make use of the coskeleton filtration on S' .

LEMMA 4.5. *There exists a spectral sequence $E_{s,k}^r \Rightarrow H_{s+k}(S'; \widehat{\mathcal{I}}')$, $d^r: E_{s,k}^r \rightarrow E_{s-r,k+r-1}^r$, $E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{H}_k \otimes \mathcal{I})$. This spectral sequence respects the inner gradings on \mathcal{I} and $\widehat{\mathcal{I}}'$*

PROOF. Consider the spectral sequence associated with the coskeleton filtration of S' for the coefficient system $\widehat{\mathcal{I}}'$:

$$E_{s,k}^r \Rightarrow H_{s+k}(S'; \widehat{\mathcal{I}}'), \quad d^r: E_{s,k}^r \rightarrow E_{s-r,k+r-1}^r, \\ E_{s,k}^1 \cong H_{s+k}(S_s, S_{s-1}; \widehat{\mathcal{I}}').$$

We have

$$E_{s,k}^1 \cong H_{s+k}(S_s, S_{s-1}; \widehat{\mathcal{I}}') = \bigoplus_{I, \dim G_I = s} H_{s+k}(G_I, \partial G_I; \widehat{\mathcal{I}}') = \bigoplus_{I, \dim G_I = s} \overline{\mathcal{I}}_k(I)$$

by Lemma 2.18. Since the values of \mathcal{I} are torsion-free, Lemma 2.19 implies

$$\bigoplus_{I, \dim G_I = s} \overline{\mathcal{I}}_k(I) \cong \bigoplus_{I, \dim G_I = s} (\mathcal{I} \otimes \mathcal{H}_k)(I) = C^{n-1-s}(S; \mathcal{I} \otimes \mathcal{H}_k).$$

Therefore,

$$E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{I} \otimes \mathcal{H}_k)$$

which proves the statement. \square

The combination of lemmas 4.2, 4.4, and 4.5 proves Theorem 3.4.

5. Manifolds with locally standard torus actions

5.1. Orbit spaces. Let T^n be a compact n -dimensional torus. The standard representation of T^n is a representation $T^n \curvearrowright \mathbb{C}^n$ by coordinate-wise rotations, i.e.

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n),$$

for $z_i, t_i \in \mathbb{C}$, $|t_i| = 1$. An action of T^n on a (compact connected smooth) manifold M^{2n} is called *locally standard*, if M has an atlas of standard charts, each isomorphic to a subset of the standard representation. More precisely, a standard chart on M is a triple (U, f, ψ) , where $U \subset M$ is a T^n -invariant open subset, ψ is an automorphism of T^n , and f is a ψ -equivariant homeomorphism $f: U \rightarrow W$ onto a T^n -invariant open subset $W \subset \mathbb{C}^n$ (i.e. $f(t \cdot y) = \psi(t) \cdot f(y)$ for all $t \in T^n$, $y \in U$).

The orbit space \mathbb{C}^n/T^n of the standard representation is the nonnegative cone $\mathbb{R}_{\geq}^n = \{x \in \mathbb{R}^n \mid x_i \geq 0\}$. Thus an orbit space of a locally standard action obtains the structure of compact connected n -dimensional manifold with corners. Recall that a manifold with corners is a topological space locally modeled by open subsets of \mathbb{R}_{\geq}^n with the combinatorial stratification induced from the face structure of \mathbb{R}_{\geq}^n (details relevant to the study of torus actions can be found in [4] or [13]).

5.2. Characteristic functions. Let $Q = M/T^n$ be the orbit space of a locally standard action. Let $\text{Fac}(Q)$ denote the set of facets (i.e. faces of codimension 1). Every face F of codimension k lies in exactly k distinct facets of Q (such manifolds with corners are called *nice* in [8] or *manifolds with faces* elsewhere). Consider the set S_Q of all faces of Q , including Q itself, and define the order on S_Q by reversed inclusion. Since Q is nice, S_Q is a simplicial poset. The minimal element of S_Q is the maximal face, that is the space Q itself. The facets of Q correspond to the vertices of S_Q . For convenience we denote abstract elements of S_Q by I, J , etc. and the corresponding faces of Q will be denoted by F_I, F_J , etc.

If $F \in \text{Fac}(Q)$ and x is a point from interior of F , then the stabilizer of x , denoted by $\lambda(F)$, is a 1-dimensional toric subgroup in T^n . If F_I is a codimension k face of Q , contained in the facets $F_1, \dots, F_k \in \text{Fac}(Q)$, then the stabilizer of an orbit $x \in F_I^\circ$ is the k -dimensional torus $T_I = \lambda(F_1) \times \dots \times \lambda(F_k) \subset T^n$, where the product is

free inside T^n . This puts a specific restriction on subgroups $\lambda(F)$, $F \in \text{Fac}(Q)$. In general, the map

$$(5.1) \quad \lambda: \text{Fac}(Q) \rightarrow \{1\text{-dimensional toric subgroups of } T^n\}$$

is called a *characteristic function*, if, whenever the facets F_1, \dots, F_k have nonempty intersection, the map

$$\lambda(F_1) \times \dots \times \lambda(F_k) \rightarrow T^n,$$

induced by inclusions $\lambda(F_i) \hookrightarrow T^n$, is injective and splits. This condition is called $(*)$ -condition. Notice, that F_1, \dots, F_k have nonempty intersection whenever the corresponding vertices of S_Q are the vertices of some simplex.

From the $(*)$ -condition follows that the map

$$(5.2) \quad H_1(\lambda(F_1) \times \dots \times \lambda(F_k); \mathbb{k}) \rightarrow H_1(T^n; \mathbb{k})$$

is also injective and splits for any ground ring \mathbb{k} . Thus the homology classes $\omega_1, \dots, \omega_k$ of subgroups $\lambda(F_1), \dots, \lambda(F_k)$ freely span a direct summand in $H_1(T^n; \mathbb{k})$. This motivates the definition of homological characteristic function given in Section 3. Surely, the exterior algebra $\Lambda[V]$ generated by a \mathbb{k} -module V has a clear meaning of the whole homology algebra of a torus: $\Lambda[H_1(T^n; \mathbb{k})] \cong H_*(T^n; \mathbb{k})$.

If the function (5.1) satisfies (5.2) for some specific ground ring \mathbb{k} , we say that λ satisfies $(*)_{\mathbb{k}}$ -condition. It is easy to see that the topological $(*)$ -condition is equivalent to $(*)_{\mathbb{Z}}$, and that $(*)_{\mathbb{Z}}$ implies $(*)_{\mathbb{k}}$ for any \mathbb{k} .

5.3. Model spaces. Let M be a manifold with locally standard action and $\mu: M \rightarrow Q$ be the projection to the orbit space. The free part of the action has the form $\mu|_{Q^\circ}: \mu^{-1}(Q^\circ) \rightarrow Q^\circ$, where $Q^\circ = Q \setminus \partial Q$ is the interior of the manifold with corners. The free part is a principal torus bundle over Q° . It can be uniquely extended over Q and defines a principal T^n -bundle $\rho: Y \rightarrow Q$.

Therefore any manifold with locally standard action determines three objects: the nice manifold with corners Q , the principal torus bundle $\rho: Y \rightarrow Q$, and the characteristic function λ . One can recover the manifold M from these data by the following standard construction.

CONSTRUCTION 5.1 (Model space). Let $\rho: Y \rightarrow Q$ be a principal T^n -bundle over a nice manifold with corners Q and λ be a characteristic function on $\text{Fac}(Q)$. Consider the space $X \stackrel{\text{def}}{=} Y / \sim$, where $y_1 \sim y_2$ if and only if $\rho(y_1) = \rho(y_2) \in F_I^\circ$ for some face F_I of Q , and y_1, y_2 lie in the same T_I -orbit of the action. There exists a natural T^n -equivariant map $f: Y \rightarrow X$.

Every manifold with locally standard torus action is equivariantly homeomorphic to its model ([13, Cor.2]), so in the following we will work with X instead of M .

5.4. Filtrations. Since Q is a manifold with corners, there is a natural filtration on Q :

$$(5.3) \quad \emptyset = Q_{-1} \subset Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} = \partial Q \subset Q = Q_n,$$

where Q_i is the union of all faces of dimension $\leq i$. It lifts to the T^n -invariant filtration on Y :

$$(5.4) \quad \emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} \subset Y_n = Y,$$

where $Y_i = \rho^{-1}(Q_i)$. This in turn descends to the filtration on X :

$$(5.5) \quad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X,$$

$X_i = f(Y_i)$. It is easily seen that (5.5) is the filtration of X by orbit types, i.e. X_i is the union of all orbits of dimension at most i . We have $\dim X_i = 2i$. The maps $\mu: X \rightarrow Q$, $\rho: Y \rightarrow Q$ and $f: Y \rightarrow X$ preserve the filtrations.

The filtrations give rise to homological spectral sequences.

$$(5.6) \quad (E_Q)_{p,q}^1 = H_{p+q}(Q_p, Q_{p-1}) \Rightarrow H_{p+q}(Q), \quad (d_Q)^r: (E_Q)_{*,*}^r \rightarrow (E_Q)_{*-r, *+r-1}^r$$

$$(5.7) \quad (E_Y)_{p,q}^1 \cong H_{p+q}(Y_p, Y_{p-1}) \Rightarrow H_{p+q}(Y), \quad (d_Y)^r: (E_Y)_{*,*}^r \rightarrow (E_Y)_{*-r, *+r-1}^r$$

$$(5.8) \quad (E_X)_{p,q}^1 \cong H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X), \quad (d_X)^r: (E_X)_{*,*}^r \rightarrow (E_X)_{*-r, *+r-1}^r.$$

In the following we also need the spectral sequence associated to the filtration of Q truncated at $Q_{n-1} = \partial Q$:

$$(5.9) \quad \emptyset = Q_{-1} \subset Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} = \partial Q,$$

$$(E_{\partial Q})_{p,q}^1 = \begin{cases} H_{p+q}(Q_p, Q_{p-1}) & \text{for } p < n, \\ 0, & \text{for } p = n \end{cases} \Rightarrow H_{p+q}(\partial Q).$$

Note that $(E_X)_{p,q}^1 = 0$ for $q > p$ by dimensional reasons. The map $f: Y \rightarrow X$ induces the map of spectral sequences

$$f_*^r: (E_Y)_{p,q}^r \rightarrow (E_X)_{p,q}^r.$$

The main topological result of this paper is the following

THEOREM 5.2. *If Q is orientable and all proper faces of Q are acyclic over \mathbb{k} , then the map $f_*^2: (E_Y)_{p,q}^2 \rightarrow (E_X)_{p,q}^2$ is an isomorphism for $q < p$ or $q = p = n$, and injective for $q = p < n$.*

6. Proof of Theorem 5.2

At first we prove a technical lemma which is extremely useful when one passes from the topology of Q to the topology of its underlying simplicial poset S_Q .

For a poset S consider a space $P = \text{Cone } |S|$. A coskeleton filtration of S extends to the coskeleton filtration of P :

$$|S_0| \subset \dots \subset |S_{n-1}| = |S| \subset P,$$

and the corresponding homological spectral sequence is denoted $(E_P)_{*,*}^*$.

For convenience we introduce the following definition.

DEFINITION 6.1. *An oriented manifold with corners Q is called Buchsbaum if all its proper faces are acyclic. If Q is Buchsbaum and Q itself is acyclic, then Q is called Cohen–Macaulay. As usual, all notions depend on the ground ring \mathbb{k} .*

Any face G of Buchsbaum manifold with corners Q is an orientable manifold with corners. The acyclicity of G implies that $H_j(G, \partial G) = 0$ for $j \neq \dim G$ and $H_{\dim G}(G, \partial G) \cong \mathbb{k}$ by Poincaré–Lefschetz duality.

LEMMA 6.2.

(1) _{n} *Let Q be a Buchsbaum manifold with corners, $\dim Q = n$, S_Q be its underlying poset, and $P = \text{Cone}(|S_Q|)$. Then there exists a face-preserving map $\varphi: Q \rightarrow P$ which induces the identity isomorphism of posets of faces and the isomorphism of the spectral sequences $\varphi_*: (E_{\partial Q})_{*,*}^r \xrightarrow{\cong} (E_S)_{*,*}^r$ for $r \geq 1$.*

(2) _{n} *If Q is Cohen–Macaulay of dimension n , then φ induces the isomorphism of spectral sequences $\varphi_*: (E_Q)_{*,*}^r \xrightarrow{\cong} (E_P)_{*,*}^r$ for $r \geq 1$.*

PROOF. A map φ is constructed inductively. 0-skeleta of Q and P are naturally identified since both correspond to the set of maximal simplices of S . There always exists an extension of φ to higher-dimensional faces since all faces of P are cones. The statement is proved by the following scheme of induction: $(2)_{\leq n-1} \Rightarrow (1)_n \Rightarrow (2)_n$. The case $n = 0$ is clear.

Let us prove the implication $(1)_n \Rightarrow (2)_n$. The map φ induces the homomorphism of the long exact sequences:

$$\begin{array}{ccccccccc} \tilde{H}_*(\partial Q) & \longrightarrow & \tilde{H}_*(Q) & \longrightarrow & H_*(Q, \partial Q) & \longrightarrow & \tilde{H}_{*-1}(\partial Q) & \longrightarrow & \tilde{H}_{*-1}(Q) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{H}_*(\partial P) & \longrightarrow & \tilde{H}_*(P) & \longrightarrow & H_*(P, \partial P) & \longrightarrow & \tilde{H}_{*-1}(\partial P) & \longrightarrow & \tilde{H}_{*-1}(P) \end{array}$$

The maps $\tilde{H}_*(Q) \rightarrow \tilde{H}_*(P)$ are isomorphisms since both groups are trivial. The maps $\tilde{H}_*(\partial Q) \rightarrow \tilde{H}_*(\partial P)$ are isomorphisms, since $(E_{\partial Q}) \xrightarrow{\cong} H_*(\partial Q)$, $(E_{\partial P}) \xrightarrow{\cong} H_*(\partial P)$ and the spectral sequences are isomorphic by (1) _{n} . Five lemma shows that $\varphi_*: (E_Q)_{n,*}^1 \rightarrow (E_P)_{n,*}^1$ is an isomorphism as well. This proves (2) _{n} .

Now we prove the implication $(2)_{\leq n-1} \Rightarrow (1)_n$. Let F_I be faces of Q and G_I faces of P . All proper faces of Q are Cohen–Macaulay of dimension $\leq n-1$. Thus $(2)_{\leq n-1}$ implies the isomorphisms $H_*(F_I, \partial F_I) \rightarrow H_*(G_I, \partial G_I)$ which sum together to the isomorphism $\varphi_*: (E_{\partial Q})_{*,*}^1 \xrightarrow{\cong} (E_{\partial P})_{*,*}^1$. \square

COROLLARY 6.3. *If Q is a Buchsbaum manifold with corners, then S_Q is Buchsbaum. Moreover, in this case S_Q is a homology manifold. If Q is Cohen–Macaulay, then S_Q is a homology sphere.*

From now on we suppose that Q is Buchsbaum, as stated in the condition of Theorem 5.2. Thus S_Q is Buchsbaum as well.

Let us return to the spaces Y and X over Q . As before, let F_I be the face of Q corresponding to $I \in S_Q$. Let $Y_I = \rho^{-1}(F_I)$ and $X_I = f(Y_I)$ be the corresponding subsets of Y and X respectively. Actually, $X_I \subset X$ is a closed submanifold of dimension $2 \dim F_I$, called a face submanifold. We set $\partial Y_I = \rho^{-1}(\partial F_I)$ and $\partial X_I = f(\partial Y_I)$ (the set ∂X_I does not have the meaning of a boundary in a topological sense, this is just a conventional notation). Note that $Y_{\hat{0}} = Y$ and $X_{\hat{0}} = X$.

We have

$$(E_Y)_{p,q}^1 \cong H_{p+q}(Y_p, Y_{p-1}) \cong \bigoplus_{|I|=n-p} H_{p+q}(Y_I, \partial Y_I)$$

$$(E_X)_{p,q}^1 \cong H_{p+q}(X_p, X_{p-1}) \cong \bigoplus_{|I|=n-p} H_{p+q}(X_I, \partial X_I)$$

REMARK 6.4. The map $f_*^1: (E_Y)_{n,q}^1 \rightarrow (E_X)_{n,q}^1$, which coincides with $f_*: H_*(Y, \partial Y) \rightarrow H_*(X, \partial X)$, is an isomorphism since the identification \sim of Construction 5.1 touches only the boundary ∂Y , thus $Y/\partial Y \cong X/\partial X$.

The space Y_I is a principal T^n -bundle over Q_I . For each $I \in S \setminus \hat{0}$, the face Q_I is acyclic. Thus there exists a trivialization $Y_I \cong Q_I \times T^n$ and we have

$$(6.1) \quad H_{p+q}(Y_I, \partial Y_I) \cong \bigoplus_{i+j=p+q} H_i(F_I, \partial F_I) \otimes H_j(T^n) \cong H_q(T^n),$$

(the groups $H_i(F_I, \partial F_I)$ vanish for $i \neq p$, and $H_p(F_I, \partial F_I) \cong \mathbb{k}$). Similarly, for X we have the identification

$$H_*(X_I, \partial X_I) \cong H_*(F_I \times T^n/T_I, \partial F_I \times T^n/T_I),$$

thus

$$(6.2) \quad H_{p+q}(X_I, \partial X_I) \cong H_q(T^n/T_I).$$

Consider the graded sheaf \mathcal{H}_q^Y on S_Q which takes the value $H_{p+q}(Y_I, \partial Y_I)$ on each $I \in S_Q$ (including $I = \hat{0}$) with the restriction maps extracted from the differential $(d_Y)^1$ similar to Construction 2.13. By (6.1), the truncated part $\underline{\mathcal{H}}_*^Y = \bigoplus_q \underline{\mathcal{H}}_q^Y$ (see Remark 2.5) is the locally constant sheaf \mathcal{L} valued by exterior algebras.

Similarly, we can define a graded sheaf \mathcal{H}_q^X on S_Q which takes the value $H_{p+q}(X_I, \partial X_I)$ on $I \in S$. Its truncated part $\underline{\mathcal{H}}_*^X = \bigoplus_q \underline{\mathcal{H}}_q^X$ is the sheaf of quotient algebras \mathcal{L}/\mathcal{I} according to (6.2). Indeed, it is easily seen that the homology algebra $H_*(T^n/T_I)$ is naturally isomorphic to the quotient of $H_*(T^n)/\mathcal{I}(I)$, where $\mathcal{I}(I)$ is the ideal generated by the subspace $H_1(T_I) \subset H_1(T^n)$.

The map $f_*^1: (E_Y)_{*,*}^1 \rightarrow (E_X)_{*,*}^1$ is equal to the map $f_*: C^*(S; \mathcal{H}_*^Y) \rightarrow C^*(S; \mathcal{H}_*^X)$. This last map coincides with $f_*: C^*(S; \mathcal{L}) \rightarrow C^*(S; \mathcal{L}/\mathcal{I})$ away from $\hat{0}$.

LEMMA 6.5. *There exists a short exact sequence of graded sheaves*

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{H}^Y \rightarrow \mathcal{H}^X \rightarrow 0.$$

PROOF. This follows from the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \underline{\mathcal{H}}^Y & \longrightarrow & \underline{\mathcal{H}}^X \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{H}^Y & \longrightarrow & \mathcal{H}^X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{H}^Y / \underline{\mathcal{H}}^Y & \xrightarrow{\cong} & \mathcal{H}^X / \underline{\mathcal{H}}^X \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

in which all vertical and two horizontal lines are exact. The lower sheaves are concentrated in $\hat{0} \in S_Q$ and the graded isomorphism between them is due to Remark 6.4. \square

Finally, the short exact sequence of Lemma 6.5 induces the long exact sequence in sheaf cohomology:

$$(6.3) \quad \rightarrow H^{i-1}(S_Q; \mathcal{I}^{(q)}) \rightarrow H^{i-1}(S_Q; \mathcal{H}_q^Y) \xrightarrow{f_*^2} H^{i-1}(S_Q; \mathcal{H}_q^X) \rightarrow H^i(S_Q; \mathcal{I}^{(q)}) \rightarrow$$

The poset S_Q is a homology manifold. Thus its structure sheaf is constant: $\mathcal{H}_0 \cong \mathbb{k}$. Corollary 3.7 implies that the groups $H^i(S_Q; \mathcal{I}^{(q)})$ vanish for $i \leq n-1-q$. From the long exact sequence (6.3) we can see that the map

$$f_*: H^{i-1}(S_Q; \mathcal{H}_q^Y) \rightarrow H^{i-1}(S_Q; \mathcal{H}_q^X)$$

is an isomorphism for $i \leq n-1-q$ and injective for $i = n-q$. This map coincides with

$$f_*^2: (E_Y)_{n-i,q}^2 \rightarrow (E_X)_{n-i,q}^2.$$

The change of indices $p = n-i$ concludes the proof of Theorem 5.2.

REMARK 6.6. Note, that the similar argument proves that the map $f^*: (E_{\partial Y})_{p,q}^2 \rightarrow (E_{\partial X})_{p,q}^2$ is an isomorphism for $p > q$ and injective for $p = q$.

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